View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Elsevier - Publisher Connector

Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
Advances in Mathematics 225 (2010) 859–881
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Graded polynomial identities and codimensions: Computing the exponential growth ☆

A. Giambruno *, D. La Mattina

Dipartimento di Matematica e Informatica, Università di Palermo, Via Archirafi 34, 90123 Palermo, Italy

Received 29 November 2009; accepted 16 March 2010

Available online 30 March 2010

Communicated by Michel Van den Bergh

Abstract

Let G be a finite abelian group and A a G-graded algebra over a field of characteristic zero. This paper is devoted to a quantitative study of the graded polynomial identities satisfied by A. We study the asymptotic behavior of $c_n^G(A)$, $n = 1, 2, \ldots$, the sequence of graded codimensions of A and we prove that if A satisfies an ordinary polynomial identity, $\lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$ exists and is an integer. We give an explicit way of computing such integer by proving that it equals the dimension of a suitable finite dimension semisimple $G \times \mathbb{Z}_2$-graded algebra related to A.

© 2010 Elsevier Inc. All rights reserved.

MSC: 16R10; 16W50; 16P90

Keywords: Graded algebra; Polynomial identity; Growth; Codimensions

1. Introduction

Let A be an algebra over a field F of characteristic zero. It is well known that the study of the polynomial identities satisfied by A is equivalent to the study of the multilinear ones and in this setting an effective way to measure such identities is through the sequence of codimensions $c_n(A)$, $n = 1, 2, \ldots$, of A. Recall that if P_n is the space of multilinear polynomials

☆ This research was partially supported by MIUR of Italy.

* Corresponding author.
E-mail addresses: a.giambruno@unipa.it (A. Giambruno), daniela.lamattina@unipa.it (D. La Mattina).

0001-8708/$ – see front matter © 2010 Elsevier Inc. All rights reserved.
doi:10.1016/j.aim.2010.03.013

in the non-commutative variables $x_1, \dots, x_n$ and $\mathrm{Id}(A)$ is the ideal of identities of A, then $c_n(A) = \dim P_n/(P_n \cap \mathrm{Id}(A))$.

The asymptotic behavior of this sequence has been extensively studied in recent years leading to classification results of several varieties of algebras. The key result in this area says that the sequence of codimensions of a PI-algebra (algebra satisfying a non-trivial polynomial identity) is exponentially bounded and its exponential rate of growth is an integer (see [14]).

When A is endowed with a structure of graded algebra the objects to study are the graded polynomial identities satisfied by A and one defines a corresponding sequence of graded codimensions. It is well known that such sequence is exponentially bounded in case of PI-algebras [10] and the link between the two sequences (ordinary and graded) is an interesting object of study. Some of the questions arising in this context are the following: can one compute the exponential rate of growth of this sequence? Is it an integer? What are the values it can assume? How is it related to the exponential rate of growth of the (ordinary) codimensions? Here we shall answer these questions when A is graded by a finite abelian group.

Throughout A will be an associative algebra over a field F of characteristic zero satisfying an ordinary polynomial identity. Assume that A is graded by a finite abelian group G and let $A = \bigoplus_{g \in G} A_g$ be the decomposition of A into the sum of its homogeneous components A_g.

Recall that a G-graded polynomial identity for A is a polynomial in non-commuting variables $x_{i,g}$, $i \geqslant 1$, $g \in G$, labeled by the group G, vanishing under any graded evaluation in A. Since the study of the G-graded identities in characteristic zero can be reduced to the study of the multilinear ones, for every $n \geqslant 1$, we define P_n^G to be the space of multilinear polynomials in the variables $x_{i,g}$, $1 \leqslant i \leqslant n$, $g \in G$. Let $\mathrm{Id}^G(A)$ be the ideal of G-graded polynomial identities satisfied by A, and define $c_n^G(A) = \dim P_n^G/(P_n^G \cap \mathrm{Id}^G(A))$, the nth graded codimension of A.

The ordinary polynomial identities and corresponding codimensions are obtained by considering the trivial grading on A (or $G = \{e\}$). Since A is a PI-algebra, by [18] the sequence $c_n(A)$, and so, $c_n^G(A)$, $n = 1, 2, \dots$, is exponentially bounded. Moreover in [11] and [12] it was shown that $\lim_{n\to\infty} \sqrt[n]{c_n(A)} = \exp(A)$ exists and is an integer called the PI-exponent of the algebra A. Such integral scale gives a way to measure the varieties of algebras and it had lead to many results including the classification of the so-called minimal varieties [3,7,13].

Our aim in this paper is to determine the exponential rate of growth of the sequence of graded codimensions. We shall prove that if A is any G-graded algebra satisfying an ordinary polynomial identity, the graded exponent $\exp^G(A) = \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$ exists and is an integer. Moreover we shall produce an explicit way of computing the graded exponent by proving that $\exp^G(A)$ equals the dimension of a suitable finite dimensional $G \times \mathbb{Z}_2$-graded algebra related to A. In case A is a G-graded finite dimensional algebra, the existence of the graded exponent was proved in [2] (when $G = \mathbb{Z}_2$ this was proved in [5]). The main result in [2] is rediscovered here as a special case. Nevertheless, a key tool used here is the existence of certain multialternating central polynomials constructed in [2].

The exponential growth of the codimensions and the integrality of the exponent for associative graded algebras is quite surprising. In fact for Lie algebras or for general non-associative algebras the situation is much more involved. For non-associative PI-algebras, the sequence of codimensions is not always exponentially bounded [17]. Moreover, even in case the codimensions are exponentially bounded, the exponential rate of growth can be non-integer. In fact in [9] the authors constructed for any real number $\alpha > 1$ an algebra whose codimensions are exponentially bounded and their exponential rate of growth is equal to α.

For Lie algebras the codimensions are not always exponentially bounded [17], but in [23] it was shown that the PI-exponent of a finite dimensional Lie algebra exists and is an integer. Nevertheless in [22] an example was given of a Lie algebra whose PI-exponent is not an integer.

The starting point of this paper is a result recently proved independently by Aljadeff and Belov [1] and Sviridova [20] asserting that the ideal of graded identities of a G-graded PI-algebra coincides with the graded identities of the Grassmann envelope of a finite dimensional $G \times \mathbb{Z}_2$-graded algebra.

The main techniques employed in this paper are methods of representation theory of the symmetric group [15] and computations of the asymptotics for the degrees of the irreducible S_n-representations [6].

2. Preliminaries

Throughout the paper F will denote a field of characteristic zero and A an associative F-algebra satisfying a non-trivial polynomial identity (PI-algebra). Let $F\langle X\rangle$ be the free associative algebra on a countable set $X = \{x_1, x_2, \dots\}$ and $\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \text{ in } A\}$ the T-ideal of (ordinary) polynomial identities of A. For every $n \geqslant 1$ we denote by $P_n = \{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ the space of multilinear polynomials in $x_1, \dots, x_n$ and $P_n(A) = P_n/(P_n \cap \mathrm{Id}(A))$. The integer $c_n(A) = \dim P_n(A)$ is called the nth codimension of A. In [18] it was proved that the sequence of codimensions is exponentially bounded and in [11] and [12] it was proved that

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

exists and is a non-negative integer called the PI-exponent of the algebra A.

Now assume that the algebra A is graded by a finite abelian group G. Let $G = \{g_1 = 1, g_2, \dots, g_s\}$ and let $A = \bigoplus_{i=1}^{s} A_{g_i}$ be the decomposition of A into its homogeneous components. Hence $A_{g_i} A_{g_j} \subseteq A_{g_i g_j}$, for all $i, j = 1, \dots, s$.

We denote by $F\langle \mathcal{X}, G\rangle$ the free associative G-graded algebra of countable rank over F. Here the set $\mathcal{X}$ decomposes as $\mathcal{X} = \bigcup_{i=1}^{s} \mathcal{X}_{g_i}$, where the sets $\mathcal{X}_{g_i} = \{x_{1,g_i}, x_{2,g_i}, \dots\}$ are disjoint, and the elements of $\mathcal{X}_{g_i}$ have homogeneous degree g_i. The algebra $F\langle \mathcal{X}, G\rangle$ has a natural G-grading $F\langle \mathcal{X}, G\rangle = \bigoplus_{g\in G} \mathcal{F}_g$, where $\mathcal{F}_g$ is the subspace of $F\langle \mathcal{X}, G\rangle$ spanned by the monomials $x_{i_1,g_{j_1}} \cdots x_{i_t,g_{j_t}}$ of homogeneous degree $g = g_{j_1} \cdots g_{j_t}$.

Recall that an element f of $F\langle \mathcal{X}, G\rangle$ is called a graded polynomial. Also, f is a graded (polynomial) identity of the algebra A, and we write $f \equiv 0$, in case f vanishes under all graded substitutions $x_{i,g} \to a_g \in A_g$.

Let $\mathrm{Id}^G(A) = \{f \in F\langle \mathcal{X}, G\rangle \mid f \equiv 0 \text{ on } A\}$ be the ideal of graded identities of A. It is easily checked that $\mathrm{Id}^G(A)$ is invariant under all graded endomorphisms of $F\langle \mathcal{X}, G\rangle$.

Notice that if for $i \geqslant 1$ we set $x_i = x_{i,g_1} + \cdots + x_{i,g_s}$, then the free algebra $F\langle X\rangle$ is naturally embedded in $F\langle \mathcal{X}, G\rangle$ and we can regard the (ordinary) identities of A as a special kind of graded identities.

Since $\mathrm{char}\, F = 0$, the multilinear polynomials of $\mathrm{Id}^G(A)$ determine all of $\mathrm{Id}^G(A)$. Hence for $n \geqslant 1$ define

$$P_n^G = \mathrm{span}_F\{x_{\sigma(1),g_{i_{\sigma(1)}}} \cdots x_{\sigma(n),g_{i_{\sigma(n)}}} \mid \sigma \in S_n, g_{i_1}, \dots, g_{i_n} \in G\}$$

to be the space of multilinear G-graded polynomials in the variables $x_{1,g_{i_1}}, \dots, x_{n,g_{i_n}}$, $g_{i_j} \in G$. Hence the ideal $\mathrm{Id}^G(A)$ is determined by the sequence of subspaces $P_n^G \cap \mathrm{Id}^G(A)$, $n = 1, 2, \dots$, and we construct the quotient spaces $P_n^G(A) = \frac{P_n^G}{P_n^G \cap \mathrm{Id}^G(A)}$. The non-negative integer

$$c_n^G(A) = \dim_F P_n^G(A), \quad n \geqslant 1,$$

is called the nth G-graded codimension of A.

The purpose of this paper is to study the asymptotic behavior of the sequence $c_n^G(A)$, $n = 1, 2, \dots$, comparing it with that of the (ordinary) codimensions $c_n(A)$, $n = 1, 2, \dots$.

In general it is easy to see that, for every n, $c_n(A) \leqslant c_n^G(A)$ and, since A is a PI-algebra, by [10] we get that

$$c_n^G(A) \leqslant |G|^n c_n(A), \quad n \geqslant 1. \tag{1}$$

Hence the sequence of G-codimensions is exponentially bounded and by [11] and [12] we immediately get that

$$\exp(A) \leqslant \liminf_{n\to\infty} \sqrt[n]{c_n^G(A)} \leqslant \limsup_{n\to\infty} \sqrt[n]{c_n^G(A)} \leqslant |G| \exp(A).$$

Here we shall prove that $\lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$ exists and we shall compute its exact value by proving that it is equal to the dimension of a suitable semisimple subalgebra of a certain finite dimensional $G \times \mathbb{Z}_2$-algebra whose Grassmann envelope satisfies the same graded identities as A (the Grassmann envelope is introduced in the sequel).

Next we are going to reduce the study of $P_n^G(A)$ to that of smaller spaces. Let $n \geqslant 1$ and write $n = n_1 + \dots + n_s$ a sum of non-negative integers. Define $P_{n_1,\dots,n_s} \subseteq P_n^G$ to be the space of multilinear graded polynomials in which the first n_1 variables have homogeneous degree g_1, the next n_2 variables have homogeneous degree n_2 and so on. Notice that given such $n_1, \dots, n_s$, there are $\binom{n}{n_1,\dots,n_s}$ subspaces isomorphic with $P_{n_1,\dots,n_s}$, where $\binom{n}{n_1,\dots,n_s} = \frac{n!}{n_1!\cdots n_s!}$ denotes the multinomial coefficient. It is clear that P_n^G is the direct sum of such subspaces with $n_1 + \dots + n_s = n$. Moreover such decomposition is inherited by $P_{n_1,\dots,n_s} \cap \mathrm{Id}^G(A)$. At the light of these observations, one defines

$$P_{n_1,\dots,n_s}(A) = \frac{P_{n_1,\dots,n_s}}{P_{n_1,\dots,n_s} \cap \mathrm{Id}^G(A)}$$

and

$$c_{n_1,\dots,n_s}(A) = \dim P_{n_1,\dots,n_s}(A).$$

Therefore, by checking dimensions we easily get that

$$c_n^G(A) = \sum_{n_1+\dots+n_s=n} \binom{n}{n_1, \dots, n_s} c_{n_1,\dots,n_s}(A). \tag{2}$$

In order to compute an upper and a lower bound for $c_n^G(A)$, our strategy will be to compute such bounds for $c_{n_1,\dots,n_s}(A)$ and then use (2).

The space $P_{n_1,\dots,n_s}(A)$ is naturally endowed with a structure of $S_{n_1} \times \cdots \times S_{n_s}$-module in the following way. The group $S_{n_1} \times \cdots \times S_{n_s}$ acts on the left on $P_{n_1,\dots,n_s}$ by permuting the variables of the same homogeneous degree; hence S_{n_1} permutes the variables of homogeneous degree g_1, S_{n_2} those of homogeneous degree g_2, etc. Since $\mathrm{Id}^G(A)$ is invariant under this action, $P_{n_1,\dots,n_s}(A)$ inherits a structure of $S_{n_1} \times \cdots \times S_{n_s}$-module and we denote by $\chi_{n_1,\dots,n_s}(A)$ its character.

If λ is a partition of n, we write $\lambda \vdash n$. It is well known that there is a one-to-one correspondence between partitions of n and irreducible S_n-characters. Hence if $\lambda \vdash n$, we denote by χ_λ the corresponding irreducible S_n-character. Now, if $\lambda(1) \vdash n_1, \dots, \lambda(s) \vdash n_s$, are partitions, then we write $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ and we say that $\langle\lambda\rangle$ is a multipartition of $n = n_1 + \cdots + n_s$.

Since $\operatorname{char} F = 0$, by complete reducibility $\chi_{n_1,\dots,n_s}(A)$ can be written as a sum of irreducible characters and let

$$\chi_{n_1,\dots,n_s}(A) = \sum_{\langle\lambda\rangle \vdash n} m_{\langle\lambda\rangle} \chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}, \tag{3}$$

where $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ is a multipartition of $n = n_1 + \cdots + n_s$ and $m_{\langle\lambda\rangle} \geqslant 0$ is the multiplicity of $\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}$ in $\chi_{n_1,\dots,n_s}(A)$. We call $\chi_{n_1,\dots,n_s}(A)$ the $(n_1, \dots, n_s)$th cocharacter of A.

A basic fact that we shall need in what follows is that the multiplicities $m_{\langle\lambda\rangle}$ in (3) are polynomially bounded.

Remark 1. (See [2, Remark 1].) There exist constants C, k such that for all $n \geqslant 1$, $m_{\langle\lambda\rangle} \leqslant Cn^k$ in (3).

Next we recall some basic facts about the representation theory of S_n. Let $\lambda \vdash n$ and let T_λ be a Young tableau of shape λ. Let R_{T_λ} and C_{T_λ} be the subgroups of S_n stabilizing the rows and the columns of T_λ, respectively. Then, if we define $R_{T_\lambda}^+ = \sum_{\sigma \in R_{T_\lambda}} \sigma$ and $C_{T_\lambda}^- = \sum_{\tau \in C_{T_\lambda}} (\operatorname{sgn} \tau)\tau$, the element $e_{T_\lambda} = R_{T_\lambda}^+ C_{T_\lambda}^-$ is an essential idempotent of FS_n generating an irreducible S_n-module whose character is χ_λ.

Now, let $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ be a multipartition of n and for every $i = 1, \dots, s$, let $T_{\lambda(i)}$ be a tableau of shape $\lambda(i)$ and $e_{T_{\lambda(i)}}$ the corresponding essential idempotent of FS_{n_i}. Let us write $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \dots, T_{\lambda(s)})$ for the multitableau on $\langle\lambda\rangle$ and let $e_{T_{\langle\lambda\rangle}} = e_{T_{\lambda(1)}} \cdots e_{T_{\lambda(s)}}$ be the corresponding essential idempotent of $F(S_{n_1} \times \cdots \times S_{n_s})$.

Lemma 1. *Let M be an $S_{n_1} \times \cdots \times S_{n_s}$-module and $e_{T_{\langle\lambda\rangle}} u \neq 0$ for some $u \in M$.*

1. *For any j and any subgroup H of $C_{T_{\lambda(j)}}$, $(\sum_{\sigma \in H} (\operatorname{sgn} \sigma)\sigma) e_{T_{\langle\lambda\rangle}} u \neq 0$.*
2. *For any j and any subgroup H of $R_{T_{\lambda(j)}}$, $(\sum_{\sigma \in H} \sigma)(C_{T_{\lambda(1)}}^- \cdots C_{T_{\lambda(s)}}^-) e_{T_{\langle\lambda\rangle}} u \neq 0$.*

Proof. Since $e_{T_{\langle\lambda\rangle}} u \neq 0$, for any j, $e_{T_{\lambda(j)}} u \neq 0$ and the result follows as in the proof of [14, Lemma 2.5.1]. Property 2 is proved similarly. □

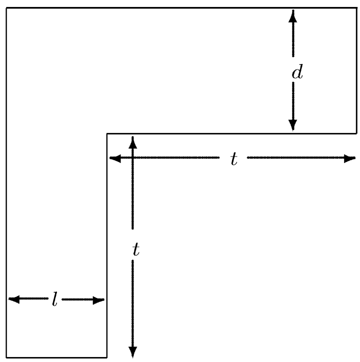
Let $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ and $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \dots, T_{\lambda(s)})$ a multitableau associated to $\langle\lambda\rangle$. We make the following

Definition 1. We say that a polynomial $f \in P_{n_1,\dots,n_s}$ corresponds to $T_{\langle\lambda\rangle}$ if $f = e_{T_{\langle\lambda\rangle}} f_0$ for some polynomial $f_0 \in P_{n_1,\dots,n_s}$.

Given integers $d, l, t \geqslant 0$ we define the partition

$$h(d,l,t) = (\underbrace{l+t,\dots,l+t}_{d}, \underbrace{l,\dots,l}_{t}).$$

Hence $h(d,l,t)$ has a hook shaped diagram as shown in the picture below



We also define an infinite hook $H(d,l)$ as follows

$$H(d,l) = \bigcup_{n\geqslant 1} \{\lambda = (\lambda_1,\dots,\lambda_r) \vdash n \mid \lambda_{d+1} \leqslant l\}.$$

If $\lambda = (\lambda_1, \lambda_2, \dots) \vdash n$ and $\mu = (\mu_1, \mu_2, \dots) \vdash m$ are two partitions, we write $\lambda \leqslant \mu$ if $\lambda_i \leqslant \mu_i$, for every $i \geqslant 1$ (the diagram of λ is contained in the diagram of μ). Accordingly, if $\langle\lambda\rangle = (\lambda(1),\dots,\lambda(s))$ and $\langle\mu\rangle = (\mu(1),\dots,\mu(s))$ are two multipartitions, we write $\langle\lambda\rangle \geqslant \langle\mu\rangle$ if for all $i = 1,\dots,s$, $\lambda(i) \geqslant \mu(i)$.

Recall that for a partition $\lambda \vdash n$, $d_\lambda = \chi_\lambda(1)$ is the degree of the irreducible S_n-character associated to λ. Here we record two results that we shall need in what follows.

Lemma 2. *(See [14, Lemma 6.2.4].) Let $\lambda \vdash n$, $\mu \vdash n'$ be such that $\mu \leqslant \lambda$. If $n - n' \leqslant c$ then $n^{-2c} d_\mu \leqslant d_\lambda \leqslant n^c d_\mu$. Hence $d_\mu \leqslant d_\lambda$.*

Lemma 3. *(See [6].) For some constants $C, r > 0$ the following inequality holds*

$$\sum_{\substack{\lambda\vdash n\\ \lambda\in H(d,l)}} d_\lambda \leqslant C n^r (d+l)^n.$$

In the following two lemmas we exploit the symmetry and skew-symmetry of the polynomials corresponding to a multitableau. If $f \in F\langle\mathcal{X}, G\rangle$ is a polynomial in the variables of the finite sets $X_{g_1},\dots,X_{g_s}$ of homogeneous degree $g_1,\dots,g_s$, respectively, we write $f = f(X_{g_1},\dots,X_{g_s})$.

Lemma 4. *Let $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s)) \vdash (n_1, \dots, n_s)$ be a multipartition of n and $f = f(X_{g_1}, \dots, X_{g_s})$ a non-zero polynomial corresponding to a multitableau $T_{\langle\lambda\rangle} = (T_{\lambda(1)}, \dots, T_{\lambda(s)})$.*

Suppose that there exists j such that $\lambda(j) \geqslant h(d, l, t)$, for some d, l, t. Then we can find a subset Y_{g_j} of X_{g_j}, which can be partitioned into a disjoint union

$$Y_{g_j} = Y^1 \cup \dots \cup Y^d,$$

$|Y^1| = \dots = |Y^d| = l + t$, with the property that for some $r \in F(S_{n_1} \times \dots \times S_{n_s})$, $rf \neq 0$ and rf is symmetric on Y^i, for all $1 \leqslant i \leqslant d$.

Moreover we can write $rf = f_1 + f_2 + \cdots$ and, for every f_i, $i \geqslant 1$, there is a partition of Y_{g_j}

$$Y_{g_j} = W^1 \cup \dots \cup W^{l+t},$$

$|W^1| = \dots = |W^{l+t}| = d$, such that f_i is alternating on W^m, $1 \leqslant m \leqslant l + t$.

Proof. Let T_j be the rectangular tableau with d rows and $t + l$ columns lying in the upper left corner of $T_{\lambda(j)}$. For $i = 1, \dots, d$, let N^i be the set of integers in the ith row of T_j and set $N = N^1 \cup \dots \cup N^d$. Also, for $k = 1, \dots, l + t$, let M^k be the set of integers contained in the kth column of T_j. Set $H = \{\sigma \in R_{T_{\lambda(j)}} \mid \sigma(i) = i$ for any $i \notin N\}$, and $r = (\sum_{\sigma\in H} \sigma)C^-_{T_{\lambda(j)}}$. Let $Y^i = \{x_{k,g_j} \mid k \in N^i\}$, $Z^i = \{x_{k,g_j} \mid k \in M^i\}$.

By Lemma 1, $rf \neq 0$. Moreover, rf is symmetric on Y^i for each i. On the other hand, the polynomial $C^-_{T_{\lambda(j)}} f$ is alternating on Z^i, therefore for any $\sigma \in H$, $\sigma C^-_{T_{\lambda(j)}} f$ is alternating in the variables of each $W^i = \sigma(Z^i)$, $1 \leqslant i \leqslant l + t$. Hence, rf is the required multilinear polynomial. □

By exchanging the role of rows and columns and their properties, we can easily get the following

Lemma 5. *Under the hypotheses of the previous lemma, there exists a subset Y_{g_j} of X_{g_j}, which can be partitioned into a disjoint union*

$$Y_{g_j} = Y^1 \cup \dots \cup Y^l,$$

$|Y^1| = \dots = |Y^l| = d + t$ with the property that for some $r \in F(S_{n_1} \times \dots \times S_{n_s})$, $rf \neq 0$ and rf is alternating on Y^i, for all $1 \leqslant i \leqslant l$.

Moreover $rf = f_1 + f_2 + \cdots$ and, for every f_i, $i \geqslant 1$, there is a partition of Y_{g_j}

$$Y_{g_j} = W^1 \cup \dots \cup W^{d+t},$$

$|W^1| = \dots = |W^{d+t}| = l$, such that f_i is symmetric on W^m, $1 \leqslant m \leqslant d + t$.

3. Guessing $\exp^G(A)$

Let $B = \bigoplus_{(g,i)\in G\times\mathbb{Z}_2} B_{(g,i)}$ be a $G \times \mathbb{Z}_2$-graded algebra. Then B has an induced $\mathbb{Z}_2$-grading, $B = B_0 \oplus B_1$, where $B_0 = \bigoplus_{g\in G} B_{(g,0)}$ and $B_1 = \bigoplus_{g\in G} B_{(g,1)}$, and an induced G-grading $B = \bigoplus_{g\in G} B_g$ where, for all $g \in G$, $B_g = B_{(g,0)} \oplus B_{(g,1)}$.

Let $E = \langle e_1, e_2, \dots \mid e_i e_j = -e_j e_i\rangle$ be the infinite dimensional Grassmann algebra over F and let $E = E_0 \oplus E_1$ be its standard $\mathbb{Z}_2$-grading. Here E_0 (resp. E_1) is the span of all monomials in the e_i's of even (resp. odd) length. Then, the Grassmann envelope of B, $E(B) = (B_0 \otimes E_0) \oplus (B_1 \otimes E_1)$ has a natural G-grading induced from the G-grading of B given by $E(B) = \bigoplus_{g\in G} E(B)_g$, where $E(B)_g = (B_{(g,0)} \otimes E_0) \oplus (B_{(g,1)} \otimes E_1)$.

In order to compute the exponential rate of growth of the graded codimensions of a G-graded algebra we need to apply a result, proved independently in [1] and [20], which extends an important theorem of Kemer [16, Theorem 2.3] to the graded case. The result is the following: let G be a finite abelian group and A a G-graded PI-algebra over a field of characteristic zero. Then there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B such that $\mathrm{Id}^G(A) = \mathrm{Id}^G(E(B))$. It is worth noticing that by [1] such result still holds if the group G is not abelian.

We fix the notation throughout the paper: $G = \{g_1, \dots, g_s\}$ is a finite abelian group and A is a finite dimensional $G \times \mathbb{Z}_2$-graded algebra over an algebraically closed field F of characteristic zero.

Now, by the Wedderburn–Malcev theorem [8], we can write

$$A = B + J$$

where B is a maximal semisimple subalgebra of A and $J = J(A)$ is its Jacobson radical. It is well known that J is a graded ideal, moreover by [21] we assume, as we may, that B is a $G \times \mathbb{Z}_2$-graded subalgebra of A. Hence we can write

$$B = B_1 \oplus \dots \oplus B_k$$

where $B_1, \dots, B_k$ are $G \times \mathbb{Z}_2$-graded simple algebras. We start by making a definition.

Definition 2. We say that a semisimple subalgebra $C = C_1 \oplus \dots \oplus C_h$, where $C_1, \dots, C_h \in \{B_1, \dots, B_k\}$ are distinct, is admissible if $C_1 J C_2 J \cdots J C_h \neq 0$.

Then we define

$$p = p(A) = \max(\dim C) \tag{4}$$

where C runs over all admissible subalgebras of B.

In Theorem 1 below we shall prove that $p(A)$ coincides with $\lim_{n\to\infty} \sqrt[n]{c_n^G(E(A))}$.

We shall make use of (2) in order to deduce an upper and lower bounds for $c_n^G(E(A))$ from an upper and lower bounds for $c_{n_1,\dots,n_s}(E(A))$.

4. Computing an upper bound for the G-codimensions

In this section we shall prove that the nth G-codimension of $E(A)$ is bounded from above by $Cn^t p^n$, for some constants C, t, where p is the integer defined in (4).

We start by proving a basic result that we shall apply in the sequel. Recall that $G = \{g_1, \dots, g_s\}$ and if B is a $G \times \mathbb{Z}_2$-graded algebra, then B inherits a structure of G-graded algebra $B = \bigoplus_{j=1}^{s} B_{g_j}$ where $B_{g_j} = B_{(g_j,0)} \oplus B_{(g_j,1)}$.

Lemma 6. *Let $A = B + J$ be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra,* $\dim A = m$, *with B a maximal $G \times \mathbb{Z}_2$-graded semisimple subalgebra. Let $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s))$ be a multipartition of n and suppose that for some j, $1 \leqslant j \leqslant s$,*

$$\lambda(j) \geqslant h(d, l, t),$$

where $d + l > \dim B_{g_j}$ and $t > (d + l)m + m$. Then any multilinear G-graded polynomial $f = e_{T_{\langle\lambda\rangle}} f_0$ corresponding to a multitableau $T_{\langle\lambda\rangle}$ vanishes in $E(A)$.

Proof. Write $B = C_1 \oplus \cdots \oplus C_k$ a sum of $G \times \mathbb{Z}_2$-graded simple algebras and fix a basis $\mathcal{A}$ of $G \times \mathbb{Z}_2$-homogeneous elements of A which is the union of corresponding bases of $C_1, \dots, C_k$ and J, respectively. In what follows all evaluations of A will be made in $\mathcal{A}$.

Let $p_0 = \dim B_{(g_j,0)}$ and $p_1 = \dim B_{(g_j,1)}$. Since $d + l > \dim B_{g_j} = \dim B_{(g_j,0)} + \dim B_{(g_j,1)}$ then either $d > \dim B_{(g_j,0)}$ or $l > \dim B_{(g_j,1)}$. Let $T_{\langle\lambda\rangle}$ be a multitableau corresponding to $\langle\lambda\rangle$ and let $f = e_{T_{\langle\lambda\rangle}} f_0$ be a non-zero corresponding polynomial. Write $f = f(X_{g_1}, \dots, X_{g_s})$.

Suppose first that $d > \dim B_{(g_j,0)}$. By Lemma 4, there exists a subset Y_{g_j} of X_{g_j}, such that

$$Y_{g_j} = Y^1 \cup \cdots \cup Y^d, \tag{5}$$

$|Y^1| = \cdots = |Y^d| = l + t$, and, for some $r \in F(S_{n_1} \times \cdots \times S_{n_s})$, $rf \neq 0$ is symmetric on Y^i, for all $1 \leqslant i \leqslant d$.

Notice that, since f generates an irreducible left $S_{n_1} \times \cdots \times S_{n_s}$-module,

$$F(S_{n_1} \times \cdots \times S_{n_s}) f = F(S_{n_1} \times \cdots \times S_{n_s}) rf$$

and, in order to prove that $f \in \mathrm{Id}^G(E(A))$, it is enough to prove that $rf \in \mathrm{Id}^G(E(A))$.

By Lemma 4, we write $rf = f_1 + f_2 + \cdots$ where each f_i is alternating on suitable disjoint subsets of Y_{g_j}. Suppose that there exists some substitution of the variables such that the corresponding evaluation of rf gives a non-zero value in $E(A)$. Then at least one of the summands f_i should have a non-zero evaluation. Let it be $f_1 \notin \mathrm{Id}^G(E(A))$. Then Y_{g_j} decomposes as

$$Y_{g_j} = W^1 \cup \cdots \cup W^{l+t},$$

where $|W^i| = d$ and f_1 is alternating on W^i, for every $1 \leqslant i \leqslant l + t$.

Notice that since f_1 is alternating on each set W^i, and our evaluations are in $\mathcal{A} \otimes E$, in order to get a non-zero value, no two variables of W^i can be evaluated into elements of the type $a \otimes e_1$ and $a \otimes e_2$ with $a \in B_{(g_j,0)}$, $e_1, e_2 \in E_0$. Thus, since $\dim B_{(g_j,0)} \leqslant d - 1$, and $|W^i| = d$, it follows that in order to get a non-zero value we must evaluate f_1 and, so rf, into at least $l + t$ elements of the type $a \otimes e$ where $a \in B_{(g_j,1)} \cup J$, $e \in E_1$ or $a \in J$, $e \in E_0$.

If $J^q = 0$, where q is the index of nilpotence of J, then at least $l + t - q + 1$ variables must be evaluated into elements of the type $a \otimes e$ with $a \in B_{(g_j,1)}$, $e \in E_1$.

Since by hypothesis $t > (d+l)m + m$, then $l + t - q + 1 > l + (d+l)m + m - q + 1 > dm$ since $\dim A = m \geqslant q$ (this is seen by considering the left regular representation of A). Therefore $l + t - q + 1 > dm$ and by looking at the decomposition (5), since $|Y^1| = \cdots = |Y^d|$, we get that there exists Y^i containing $m+1$ variables that must be evaluated into elements of the type $a \otimes e$ with $a \in B_{(g_j,1)}$, $e \in E_1$. Since $m + 1 > \dim B_{(g_j,1)}$ it follows that at least two variables in Y^i take value $c \otimes e_1$ and $c \otimes e_2$, where $c \in B_{(g_j,1)} \cap \mathcal{A}$, $e_1, e_2 \in E_1$. But rf is symmetric on Y^i. Hence, the corresponding value will be zero, a contradiction.

Now assume that $l > \dim B_{(g_j,1)}$.

By Lemma 5, the set Y_{g_j} decomposes as

$$Y_{g_j} = Y^1 \cup \cdots \cup Y^l,$$

$|Y^1| = \cdots = |Y^l| = d + t$ and for some $r' \in F(S_{n_1} \times \cdots \times S_{n_s})$, $r'f$ is alternating on Y^i, for all $1 \leqslant i \leqslant l$. Moreover $r'f = f_1 + f_2 + \cdots$, and each f_i is symmetric on suitable disjoint subsets of Y_{g_j}.

Suppose that $r'f \notin \mathrm{Id}^G(E(A))$. In particular, there exists a summand with non-zero evaluation. Let it be f_1. Then Y_{g_j} decomposes as

$$Y_{g_j} = W^1 \cup \cdots \cup W^{d+t},$$

where $|W^i| = l$ and f_1 is symmetric on W^i, $1 \leqslant i \leqslant d+t$.

Since f_1 is symmetric on each W^i, in order to get a non-zero evaluation, no two variables of the same W^i can be evaluated into $a \otimes e_1$, $a \otimes e_2$, $a \in B_{(g_j,1)} \cap \mathcal{A}$, $e_1, e_2 \in E_1$. Thus, since $\dim B_{(g_j,1)} \leqslant l - 1$ and $|W^i| = l$, in order to get a non-zero value, we must evaluate f_1 into at least $d + t$ elements of the type $a \otimes e$ where $a \in (B_{(g_j,0)} \cup J) \cap \mathcal{A}$, $e \in E_0$ or $a \in J \cap \mathcal{A}$, $e \in E_1$. Since $J^q = 0$, we have to replace at least $d + t - q + 1 > lm$ variables of Y_{g_j} with elements of the type $a \otimes e$, with $a \in B_{(g_j,0)} \cap \mathcal{A}$, $e \in E_0$.

Since $\dim B_{(g_j,0)} \leqslant m$, we should replace at least two variables of the same alternating set Y^i with tensors $c \otimes e_1$, $c \otimes e_2$ where $e_1, e_2 \in E_0$ and $c \in B_{(g_j,0)} \cap \mathcal{A}$. But in this case $r'f$ will take a zero value and the proof is complete. □

A consequence of the above lemma is the following.

Corollary 1. *Let $A = B + J$ be as in the previous lemma,* $\dim A = m$. *Let $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s))$ be such that, for some j, $1 \leqslant j \leqslant s$,*

$$\lambda(j) \geqslant h\big(d, l, (m+1)^2\big),$$

with $d + l =$ $\dim B_{g_j} + 1$. *Then any multilinear G-graded polynomial corresponding to a multitableau $T_{\langle\lambda\rangle}$ vanishes in $E(A)$.*

Proof. Since $d + l = \dim B_{g_j} + 1$ and $m \geqslant \dim B_{g_j}$, we get $(m+1)^2 > (m+1)m + m \geqslant (\dim B_{g_j} + 1)m + m = (d+l)m + m$ and the result follows from the previous lemma with $t = (m+1)^2$. □

We are now in a position to determine a useful upper bound for $c_n^G(E(A))$.

Lemma 7. *Let A be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra and let p be the integer defined in* (4). *Then there exist constants C_1, r_1 such that $c_n^G(E(A)) \leqslant C_1 n^{r_1} p^n$.*

Proof. If A is a nilpotent algebra the conclusion of the lemma is clearly true. Hence we may assume that $A = B + J$, $B \neq 0$ and $B = B_1 \oplus \cdots \oplus B_k$, where $B_1, \ldots, B_k$ are $G \times \mathbb{Z}_2$-graded simple algebras. Fix a basis $\mathcal{A}$ of $G \times \mathbb{Z}_2$-homogeneous elements of A which is the union of corresponding bases of $B_1, \ldots, B_k$ and J, respectively. In what follows all evaluations of A will be made in $\mathcal{A}$.

Let $J^q = 0$ and consider any $n > q$. For a multipartition $\langle\lambda\rangle = (\lambda(1), \ldots, \lambda(s)) \vdash (n_1, \ldots, n_s)$ of n, let $\chi_{n_1,\ldots,n_s}(E(A))$ be the $(n_1, \ldots, n_s)$th cocharacter of $E(A)$ and write

$$\chi_{n_1,\ldots,n_s}\big(E(A)\big) = \sum_{\langle\lambda\rangle\vdash n} m_{\langle\lambda\rangle}\chi_{\lambda(1)} \otimes \cdots \otimes \chi_{\lambda(s)}. \tag{6}$$

Let $f = e_{T_{\langle\lambda\rangle}} f_0$ be a polynomial corresponding to $\langle\lambda\rangle$ and suppose that $f \notin \mathrm{Id}^G(E(A))$. Let $a_1, \ldots, a_r \in \mathcal{A}$ be such that $f(a_1 \otimes e_1, \ldots, a_r \otimes e_r) \neq 0$, for suitable $e_1, \ldots, e_r \in E$. Consider the simple components B_i containing at least one of these elements. Suppose for simplicity that they are $B_1, \ldots, B_r$. Let $B' = B_1 \oplus \cdots \oplus B_r$ and set $A' = B' + J$. Since $n > q$, $B' \neq 0$. Now, any monomial of f belongs to $B_{i_1} J \cdots J B_{i_r} \neq 0$, for some permutation $\binom{1 \ \cdots \ r}{i_1 \ \cdots \ i_r}$. It follows that $f \notin \mathrm{Id}^G(E(A'))$. Also, by the definition of p, $\dim(B') \leqslant p$. Hence, if we set $p_j = \dim B'_{g_j}$, $1 \leqslant j \leqslant s$, we have that $\sum_j p_j \leqslant p$. Let $\dim A' = m$. Now, if for some $j \in \{1, \ldots, s\}$, $\lambda(j) \geqslant h(d, l, t)$, where $d + l = p_j + 1$ and $t = (m+1)^2$, by the previous corollary we get that $f \in \mathrm{Id}^G(E(A'))$, a contradiction.

We claim that for every $j \in \{1, \ldots, s\}$, $\lambda(j)$ is contained in $H(d, p_j - d) \cup (s^u)$, the union of an infinite hook $H(d, p_j - d)$, $d \geqslant 0$, and a rectangular diagram (s^u) where $s = (m+1)^2 + m - p_j + d$ and $u = (m+1)^2 + m - d$. In other words $\lambda(j) \leqslant H(d, p_j - d) \cup (s^u)$ where

$$H(d, p_j - d) \cup \big(s^u\big) = \bigcup_{n\geqslant 1} \big\{\lambda = (\lambda_1, \ldots, \lambda_r) \vdash n \;\big|\; \lambda_{d+1} \leqslant p_j - d + s,\ \lambda_{d+u+1} \leqslant p_j - d\big\}.$$

In fact, write $\lambda(j) = (\lambda_1, \lambda_2, \ldots)$ and suppose that for some i, $\lambda_i > (m+1)^2 + m$. Let k be the integer such that $\lambda_k > (m+1)^2 + m$ and $\lambda_{k+1} \leqslant (m+1)^2 + m$. If $k > p_j$, then $\lambda \geqslant h(p_j + 1, 0, (m+1)^2)$ and we reach a contradiction by Corollary 1. Thus $k \leqslant p_j$.

Set $(m+1)^2 + m + 1 = t$. If $\lambda_t \geqslant p_j - k + 1$, then $\lambda(j) \geqslant \mu$ where $\mu = (\mu_1, \ldots, \mu_t)$ is such that $\mu_1 = \cdots = \mu_k = t$ and $\mu_{k+1} = \cdots = \mu_t = p_j - k + 1$, i.e., $\mu = (t^k, (p_j - k + 1)^{t-k})$.

Since $(m+1)^2 + m + 1 - (p_j - k + 1) \geqslant (m+1)^2$ and $(m+1)^2 + m + 1 - k \geqslant (m+1)^2$, we see that $\mu \geqslant h(k, p_j - k + 1, (m+1)^2)$. Hence $\lambda(j) \geqslant h(k, p_j - k + 1, (m+1)^2)$ and again we get a contradiction by Corollary 1. Thus $\lambda_t \leqslant p_j - k$ and $\lambda(j)$ is contained in $H(k, p_j - k) \cup (s^u)$ as wished.

Therefore we may assume that $\lambda_1 \leqslant (m+1)^2 + m$. Clearly $\lambda_t \leqslant p_j$ since otherwise $\lambda(j) \geqslant h(0, p_j + 1, (m+1)^2)$, contrary to Corollary 1. This says that $\lambda(j)$ is contained in $H(0, p_j) \cup (s^u)$ and the claim is proved.

The claim just proved implies that for all $j \in \{1, \ldots, s\}$, $\lambda(j)$ contains a subpartition $\mu(j)$ such that $\mu(j) \leqslant H(d, p_j - d)$ and, if $\lambda(j) \vdash n_j$, $\mu(j) \vdash n'_j$ we have that $n_j - n'_j \leqslant T_j = su$. By Lemma 2, $d_{\lambda(j)} \leqslant n_j^{T_j} d_{\mu(j)}$.

The property just proved says that if $m_{\langle\lambda\rangle} \neq 0$, there exists a multipartition $\langle\mu\rangle = (\mu(1), \dots, \mu(s)) \vdash (n'_1, \dots n'_s)$, $n'_1 + \cdots + n'_s = n'$, such that $\lambda(j) \geqslant \mu(j)$, $\mu(j) \leqslant H(d, p_j - d)$, for some d, and $d_{\lambda(j)} \leqslant n_j^T d_{\mu(j)} \leqslant n^T d_{\mu(j)}$, for $1 \leqslant i \leqslant s$ where $T = (m+1)^6$.

For $n = n_1 + \cdots + n_s$, define

$$S = S_{n_1,\dots,n_s} = \bigl\{\langle\lambda\rangle = \bigl(\lambda(1), \dots, \lambda(s)\bigr) \vdash (n_1, \dots, n_s) \bigm| m_{\langle\lambda\rangle} \neq 0 \text{ in (6)}\bigr\}.$$

By Remark 1, the multiplicities in (6) are polynomially bounded, hence there exists a constant $k > 0$ such that $m_{\langle\lambda\rangle} \leqslant n^k$, for all $\langle\lambda\rangle \in S$. By using Lemmas 2 and 3 we obtain

$$\begin{aligned} c_{n_1,\dots,n_s}\bigl(E(A)\bigr) &= \sum_{\substack{\langle\lambda\rangle\in S\\ \langle\lambda\rangle=(\lambda(1),\dots,\lambda(s))}} m_{\langle\lambda\rangle} d_{\lambda(1)} \cdots d_{\lambda(s)} \\ &\leqslant n^k n^{sT} \sum_{j=1}^{s} \sum_{i_j=0}^{p_j} \sum_{\substack{\mu(j)\vdash n'_j \leqslant n_j\\ \mu(j)\in H(i_j, p_j - i_j)}} d_{\mu(1)} \cdots d_{\mu(s)} \leqslant C_1 n^a p_1^{n_1} \cdots p_s^{n_s}, \end{aligned}$$

for some constants C_1, a. By the multinomial theorem we obtain

$$\begin{aligned} c_n^G\bigl(E(A)\bigr) &= \sum_{n_1+\cdots+n_s=n} \binom{n}{n_1, \dots, n_s} c_{n_1,\dots,n_s}\bigl(E(A)\bigr) \\ &\leqslant C_1 n^a \sum_{n_1+\cdots+n_s=n} \binom{n}{n_1, \dots, n_s} p_1^{n_1} \cdots p_s^{n_s} \leqslant C_1 n^a (p_1 + \cdots + p_s)^n = C_1 n^a p^n \end{aligned}$$

and the proof is complete. □

5. $G \times \mathbb{Z}_2$-graded simple algebras

We introduce the notion of multialternating polynomial that will be an essential tool throughout the paper.

Let $t > 0$ be an integer and let

$$X_{g_i}^j = \bigl\{x_{1,g_i}^j, \dots, x_{m_i,g_i}^j\bigr\} \subseteq \mathcal{X}_{g_i}, \quad 1 \leqslant j \leqslant t,\ 1 \leqslant i \leqslant s,$$

be ts distinct sets of homogeneous variables, $|X_{g_i}^j| = m_i$. Let also $W \subseteq \bigcup_{i=1}^s \mathcal{X}_{g_i}$ be another set of homogeneous variables disjoint from the previous sets.

Let $f = f(X_{g_1}^1, \dots, X_{g_s}^1, \dots, X_{g_1}^t, \dots, X_{g_s}^t, W) \in F\langle\mathcal{X}, G\rangle$ be a multilinear graded polynomial in the variables of the sets $X_{g_i}^j$ and W, $1 \leqslant i \leqslant s$ and $1 \leqslant j \leqslant t$.

Definition 3. If f is alternating in the indeterminates of each set $X_{g_i}^j$, then we say that f is t-fold $(m_1, \dots, m_s)$-alternating. In case $t = 1$ we simply say that f is $(m_1, \dots, m_s)$-alternating.

In order to simplify the notation, in the free $G \times \mathbb{Z}_2$-graded algebra $F\langle \mathcal{X}, G \times \mathbb{Z}_2\rangle$ we shall write $x_{i,(g,0)} = y_{i,g}$ and $x_{i,(g,1)} = z_{i,g}$, for $g \in G$ and $i \geqslant 1$. Accordingly for homogeneous sets of variables we shall write $X_{(g,0)} = Y_g$ and $X_{(g,1)} = Z_g$.

In case a polynomial f is t-fold $(m_1, \dots, m_s)$-alternating on variables of homogeneous degree $(g, 0)$, $g \in G$, i.e., variables of the sets $Y^j_{g_i}$, we say that f is t-fold $(m_1, \dots, m_s)$-alternating on Y. Similarly we define polynomials t-fold $(m_1, \dots, m_s)$-alternating on Z.

Let $f \in F\langle \mathcal{X}, G \times \mathbb{Z}_2\rangle$ be a multilinear polynomial and write f in the form

$$f = \sum_{\substack{\sigma \in S_m \\ W=(w_0,w_1,\dots,w_m)}} \alpha_{\sigma,W} w_0 z_{\sigma(1)} w_1 \cdots w_{m-1} z_{\sigma(m)} w_m$$

where $z_1, \dots, z_m$ are variables of homogeneous degree $(g, 1)$, $g \in G$, $w_0, w_1, \dots, w_m$ are monomials in variables of homogeneous degree $(g, 0)$, $g \in G$ and $\alpha_{\sigma,W} \in F$. Then define

$$\tilde{f} = \sum_{\substack{\sigma \in S_m \\ W=(w_0,w_1,\dots,w_m)}} (\operatorname{sgn}\sigma) \alpha_{\sigma,W} w_0 z_{\sigma(1)} w_1 \cdots w_{m-1} z_{\sigma(m)} w_m.$$

Recall that according to [14], the map $\tilde{}$ has the following basic properties:

1) f is a $G \times \mathbb{Z}_2$-graded identity of $E(A)$ if and only if $\tilde{f}$ is a $G \times \mathbb{Z}_2$-graded identity of A;
2) $\tilde{\tilde{f}} = f$;
3) for any subset of variables Z' of $\{z_1, \dots, z_m\}$, f is alternating on Z' if and only if $\tilde{f}$ is symmetric on Z'.

Lemma 8. *Let B be a $G \times \mathbb{Z}_2$-graded algebra over F and let $d_i = \dim B_{(g_i,0)}$, $l_i = \dim B_{(g_i,1)}$, $1 \leqslant i \leqslant s = |G|$. Let $t, r \geqslant 0$ be integers and suppose that $f \in F\langle \mathcal{X}, G \times \mathbb{Z}_2\rangle$ is a multilinear polynomial t-fold $(d_1, \dots, d_s)$-alternating on Y and r-fold $(l_1, \dots, l_s)$-symmetric on Z. If $f \notin \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$, then there exist multipartitions $\langle\lambda\rangle = ((t^{d_1}), \dots, (t^{d_s}))$, $\langle\mu\rangle = ((l_1{}^r), \dots, (l_s{}^r))$, and corresponding multitableaux $T_{\langle\lambda\rangle}$, $T_{\langle\mu\rangle}$ such that $e_{T_{\langle\lambda\rangle}} e_{T_{\langle\mu\rangle}} f \notin \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$.*

Proof. Let $n = \deg f$ and P_n the space of multilinear polynomials spanned only by all monomials in the variables appearing in f. For a fixed i, $1 \leqslant i \leqslant s$, we let the symmetric group $S_{d_i t}$ act on P_n by permuting the variables in $Y^1_{g_i} \cup \cdots \cup Y^t_{g_i}$. We consider $FS_{d_i t} f$, the left $S_{d_i t}$-module generated by f, and its decomposition into irreducibles. Since $f \notin \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$, there exists a partition $\lambda(i) = (\lambda_1, \dots, \lambda_u) \vdash d_i t$ and a Young tableau $T_{\lambda(i)}$, such that $e_{T_{\lambda(i)}} f \notin \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$.

We claim that $\lambda_1 \leqslant t$ and $u \leqslant d_i$, i.e., the first row and the first column of $T_{\lambda(i)}$ are bounded by t and d_i, respectively.

In fact, if $\lambda_1 \geqslant t + 1$ then $e_{T_{\lambda(i)}} f$ is symmetric on at least $t + 1$ variables of the set $Y^1_{g_i} \cup \cdots \cup Y^t_{g_i}$. But f is alternating on each of the disjoint sets $Y^j_{g_i}$. Therefore $e_{T_{\lambda(i)}} f = 0$ is the zero polynomial being simultaneously symmetric and alternating in at least two variables.

Now suppose that $u \geqslant d_i + 1$ and write $e_{T_{\lambda(i)}} = R^+_{T_{\lambda(i)}} C^-_{T_{\lambda(i)}}$. Since $u \geqslant d_i + 1$, the polynomial $C^-_{T_{\lambda(i)}} f$ is alternating on some $d_i + 1$ variables of the set $Y^1_{g_i} \cup \cdots \cup Y^t_{g_i}$. But then also the polynomial $C^-_{T_{\lambda(i)}} \tilde{f}$ is alternating on the same variables. Since $d_i = \dim B_{(g_i,0)}$ it follows that

$C^-_{T_{\lambda(i)}}\tilde{f} \in \mathrm{Id}^{G\times\mathbb{Z}_2}(B)$ and, so, also $e_{T_{\lambda(i)}}\tilde{f} \in \mathrm{Id}^{G\times\mathbb{Z}_2}(B)$. By the first property of the map $\tilde{}$ given above, we get that $\widetilde{e_{T_{\lambda(i)}}\tilde{f}} = e_{T_{\lambda(i)}}f \in \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$. This proves the claim.

As an outcome we get that $\lambda(i) = (t^{d_i})$ is a rectangle with d_i rows and t columns. Now, since for $1 \leqslant i < j \leqslant s$, the groups $S_{d_i t}$ and $S_{d_j t}$ act on disjoint sets of variables appearing in the monomials of P_n, we deduce that there exists a multipartition $\langle\lambda\rangle = ((t^{d_1}), \dots, (t^{d_s}))$ and corresponding multitableaux $T_{\langle\lambda\rangle}$ such that $e_{T_{\langle\lambda\rangle}}f \notin \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$.

Now we let the symmetric group $S_{l_i r}$ act on P_n by permuting the variables in $Z^1_{g_i} \cup \dots \cup Z^r_{g_i}$. Let $\varphi = e_{T_{\langle\lambda\rangle}}f$ and, as above we let $FS_{l_i r}\varphi$ be the left $S_{l_i r}$-module generated by φ, and consider its decomposition into irreducibles. Since $\varphi \notin \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$, there exists a partition $\mu(i) = (\mu_1, \dots, \mu_v) \vdash l_i r$ and a Young tableau $T_{\mu(i)}$, such that $e_{T_{\mu(i)}}\varphi \notin \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$.

We claim that $\mu_1 \leqslant l_i$ and $v \leqslant r$. Suppose first that $\mu_1 \geqslant l_i + 1$. Then the polynomial $e_{T_{\mu(i)}}\varphi$ is symmetric on $l_i + 1$ variables belonging to the set $Z^1_{g_i} \cup \dots \cup Z^r_{g_i}$. By the properties of the map $\tilde{}$, the polynomial $\widetilde{e_{T_{\mu(i)}}\varphi}$ is alternating on the same $l_i + 1$ graded variables and, since $l_i = \dim B_{(g_i,1)}$, it follows that $\widetilde{e_{T_{\mu(i)}}\varphi} \in \mathrm{Id}^{G\times\mathbb{Z}_2}(B)$. But then by applying the map $\tilde{}$ again we deduce that $e_{T_{\mu(i)}}\varphi = \widetilde{\widetilde{e_{T_{\mu(i)}}\varphi}} \in \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$.

Now let $v \geqslant r + 1$. Then the polynomial $C^-_{T_{\mu(i)}}\varphi$ is alternating on at least $r+1$ variables of the set $Z^1_{g_i} \cup \dots \cup Z^r_{g_i}$. But φ is symmetric on each of the r disjoint sets $Z^j_{g_i}$. Therefore $C^-_{T_{\mu(i)}}\varphi = 0$ is the zero polynomial being simultaneously symmetric and alternating in at least two variables. Therefore also $e_{T_{\mu(i)}}\varphi = 0$. This proves the claim and we get that $\mu(i) = (l_i^r)$ is a rectangle with r rows and l_i columns.

As above we deduce that there exists a multipartition $\langle\mu\rangle = (({l_1}^r), \dots, ({l_s}^r))$, and corresponding multitableaux $T_{\langle\mu\rangle}$ such that $e_{T_{\langle\mu\rangle}}\varphi = e_{T_{\langle\lambda\rangle}}e_{T_{\langle\mu\rangle}}f \notin \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$. This completes the proof. $\square$

Lemma 9. *Let B be a finite dimensional $G \times \mathbb{Z}_2$-graded simple algebra over an algebraically closed field F. Let* $\dim B_{(g_i,0)} = p_i$, $\dim B_{(g_i,1)} = q_i$, $1 \leqslant i \leqslant s$. *Then, for any positive integer t, there exist a multipartition $\langle\lambda\rangle = (\lambda(1), \dots, \lambda(s))$ with*

$$h(p_i, q_i, 2t - \dim B) \leqslant \lambda(i) \leqslant h(p_i, q_i, 2t),$$

$1 \leqslant i \leqslant s$, and a multitableau $T_{\langle\lambda\rangle}$ such that $E(B)$ does not satisfy any G-graded identity corresponding to $T_{\langle\lambda\rangle}$.

Proof. According to [2, Lemma 18], for every $t \geqslant 1$, there exists a $2t$-fold $(p_1, q_1, \dots, p_s, q_s)$-alternating polynomial

$$\varphi = \varphi\big(Y^1_{g_1}, Z^1_{g_1}, \dots, Y^1_{g_s}, Z^1_{g_s}, \dots, Y^{2t}_{g_1}, Z^{2t}_{g_1}, \dots, Y^{2t}_{g_s}, Z^{2t}_{g_s}\big),$$

$|Y^j_{g_i}| = p_i$, $|Z^j_{g_i}| = q_i$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant 2t$, which is not a $G \times \mathbb{Z}_2$-graded identity of B and takes an invertible central value in B. By the basic properties of the map $\tilde{}$, the polynomial $\tilde{\varphi}$ is $2t$-fold $(p_1, \dots, p_s)$-alternating and $2t$-fold $(q_1, \dots, q_s)$-symmetric. Moreover $\tilde{\varphi}$ is not a $G \times \mathbb{Z}_2$-graded identity of $E(B)$.

By Lemma 8 there exist multipartitions $\langle\lambda\rangle = (((2t)^{p_1}), \dots, ((2t)^{p_s}))$, $\langle\mu\rangle = (({q_1}^{(2t)}), \dots, ({q_s}^{(2t)}))$, and corresponding multitableaux $T_{\langle\lambda\rangle}, T_{\langle\mu\rangle}$ such that

$$\psi = e_{T_{\langle\lambda\rangle}} e_{T_{\langle\mu\rangle}} \tilde{\varphi} \notin \mathrm{Id}^{G\times\mathbb{Z}_2}\big(E(B)\big).$$

Let W_n the space of multilinear polynomials spanned by the monomials in the $2t(p_1 + \cdots + p_s + q_1 + \cdots + q_s)$ variables appearing in ψ. For any i, $1 \leqslant i \leqslant s$, we let the symmetric groups S_{2tp_i} and S_{2tq_i} act on W_n by permuting the variables in $Y^1_{g_i} \cup \cdots \cup Y^{2t}_{g_i}$ and $Z^1_{g_i} \cup \cdots \cup Z^{2t}_{g_i}$, respectively. Let M be the $S_{2tp_1} \times S_{2tq_1} \times \cdots \times S_{2tp_s} \times S_{2tq_s}$-module generated by ψ and let

$$\bar{M} = M \uparrow^{S_{2t(p_1+q_1)}\times\cdots\times S_{2t(p_s+q_s)}}$$

be the induced module. Notice that since $M \nsubseteq \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$, then $\bar{M} \nsubseteq \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$.

We consider the decomposition $\bar{M} = \bar{M}_1 \oplus \cdots \oplus \bar{M}_r$ where $\bar{M}_1, \ldots, \bar{M}_r$ are irreducible $S_{2t(p_1+q_1)} \times \cdots \times S_{2t(p_s+q_s)}$-submodules. According to the Littlewood–Richardson rule [15] applied to every symmetric group, each $\bar{M}_j$ is associated to a multipartition $\nu = (\nu(1), \ldots, \nu(s))$ such that for every $i = 1, \ldots, s$,

$$h(p_i, q_i, 2t - s_i) \leqslant \nu(i) \leqslant h(p_i, q_i, 2t)$$

where $s_i = \max\{p_i, q_i\}$. Hence $\nu(i) \geqslant h(p_i, q_i, 2t - \dim B)$.

Since $\bar{M} \nsubseteq \mathrm{Id}^{G\times\mathbb{Z}_2}(E(B))$ it follows that for some multilinear $u \in \bar{M}$ and some multitableau $T_{\langle\nu\rangle}$, we must have that $e_{T_{\langle\nu\rangle}} u$ is not a $G \times \mathbb{Z}_2$-graded identity of $E(B)$. If we now rename the sets of variables $Y^j_{g_i} \cup Z^j_{g_i} = X^j_{g_i}$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant 2t$, then the corresponding polynomial $e_{T_{\langle\nu\rangle}} u$ is not a G-graded identity of $E(B)$ and the proof is complete. $\square$

6. Computing the lower bound

Let B be a finite dimensional $G \times \mathbb{Z}_2$-graded simple algebra over an algebraically closed field F. Then, according to [4] B has the following structure: there exist a subgroup H of $G \times \mathbb{Z}_2$, a 2-cocycle $f : H \times H \to F^*$ where the action of H on F is trivial, an integer k and a k-tuple $(a_1 = 1, a_2, \ldots, a_k) \in (G \times \mathbb{Z}_2)^k$ such that B is $G \times \mathbb{Z}_2$-graded isomorphic to $C = F^f H \otimes M_k(F)$ where $C_a = \mathrm{span}_F\{b_h \otimes e_{ij} : a = a_i^{-1} h a_j\}$. Here $b_h \in F^f H$ is a representative of $h \in H$ and the e_{ij}'s are the matrix units of $M_k(F)$.

Lemma 10. *If B is a finite dimensional $G \times \mathbb{Z}_2$-graded simple algebra over an algebraically closed field F, then for any non-zero homogeneous elements $b_1, b_2 \in B$ there exist homogeneous elements $c_1, c_2 \in B$ such that $c_1 b_1 c_2 = b_2$.*

Proof. If $b_1 = b_{h_1} \otimes e_{ij}$ and $b_2 = b_{h_2} \otimes e_{kl}$, we can take $c_1 = b_{{h_1}^{-1}} \otimes e_{ki}$ and $c_2 = f({h_1}^{-1}, h_1)^{-1} f(1,1)^{-1} b_{h_2} \otimes e_{jl}$, where $f : H \times H \to F^*$ is the above 2-cocycle. $\square$

Lemma 11. *Let $A = B_1 \oplus \cdots \oplus B_k + J$ be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra over an algebraically closed field F and suppose that $B_1 J B_2 J \cdots J B_r \neq 0$, for some distinct $G \times \mathbb{Z}_2$-graded simple algebras $B_1, \ldots, B_r$. Let $f_1, \ldots, f_r$ be multilinear G-graded polynomials on distinct sets of variables such that for every $i = 1, \ldots, r$, $f_i \notin \mathrm{Id}^G(E(B_i))$. Then the multilinear polynomial*

$$u_1 f_1 v_1 w_1 u_2 f_2 v_2 w_2 \cdots w_{r-1} u_r f_r v_r, \tag{7}$$

where u_1, v_1, $w_1, \dots, w_{r-1}$, u_r, v_r are new homogeneous variables, is not a G-graded identity of $E(A)$.

Proof. Since by hypothesis $B_1 J B_2 J \cdots J B_r \neq 0$, there exist homogeneous elements (in the $G \times \mathbb{Z}_2$-grading) $b_1 \in B_1, \dots, b_r \in B_r$, $j_1, \dots, j_{r-1} \in J$ such that

$$b_1 j_1 b_2 j_2 \cdots j_{r-1} b_r \neq 0. \tag{8}$$

Let $f_i = f_i(x_1^i, \dots, x_{n_i}^i)$. Since f_i is not a G-graded identity of $E(B_i)$, there exist homogeneous elements in the $G \times \mathbb{Z}_2$-grading $\bar{x}_1^i, \dots, \bar{x}_{n_i}^i \in B_i$, $e_1^i, \dots, e_{n_i}^i \in E$ such that $f_i(\bar{x}_1^i \otimes e_1^i, \dots, \bar{x}_{n_i}^i \otimes e_{n_i}^i) \neq 0$.

Now, for $t = 1, \dots, n_i$, if for some $g \in G$, $\bar{x}_t^i \in (B_i)_{(g,0)}$ (resp. $\bar{x}_t^i \in (B_i)_{(g,1)}$), we set $x_t^i = y_t^i$ (resp. $x_t^i = z_t^i$), a variable of homogeneous degree $(g, 0)$ (resp. $(g, 1)$). In this way we can regard f_i as a $G \times \mathbb{Z}_2$-graded polynomial and, by the property of $\tilde{}$, we can write

$$f_i\big(\bar{x}_1^i \otimes e_1^i, \dots, \bar{x}_{n_i}^i \otimes e_{n_i}^i\big) = \tilde{f}_i\big(\bar{x}_1^i, \dots, \bar{x}_{n_i}^i\big) \otimes e_1^i \cdots e_{n_i}^i.$$

It is obvious that $\tilde{f}_i(\bar{x}_1^i, \dots, \bar{x}_{n_i}^i) = b_i' \neq 0$, for some homogeneous (in the $G \times \mathbb{Z}_2$-grading) element $b_i' \in B_i$. By Lemma 10 one can choose homogeneous elements $a_i, c_i \in B_i$ such that $a_i b_i' c_i = b_i$. Therefore the polynomial $u_i \tilde{f}_i v_i$ takes the value b_i by evaluating $u_i, x_1^i, \dots, x_{n_i}^i, v_i$ in $a_i, \bar{x}_1^i, \dots, \bar{x}_{n_i}^i, c_i$, respectively. Let h_i, h_i', t_i be elements of E of the same homogeneous degree (in the $\mathbb{Z}_2$-grading) as a_i, c_i, j_i, respectively. Then for $i = 1, \dots, r-1$, we get

$$\begin{aligned}
&(a_i \otimes h_i) f_i\big(\bar{x}_1^i \otimes e_1^i, \dots, \bar{x}_{n_i}^i \otimes e_{n_i}^i\big)\big(c_i \otimes h_i'\big)(j_i \otimes t_i) \\
&\quad = a_i \tilde{f}_i\big(\bar{x}_1^i, \dots, \bar{x}_{n_i}\big) c_i j_i \otimes h_i e_1^i \cdots e_{n_i}^i h_i' t_i = b_i j_i \otimes h_i e_1^i \cdots e_{n_i}^i h_i' t_i.
\end{aligned}$$

For $i = r$ similarly we get

$$(a_r \otimes h_r) f_r\big(\bar{x}_1^r \otimes e_1^r, \dots, \bar{x}_{n_r}^r \otimes e_{n_r}^r\big)\big(c_r \otimes h_r'\big) = b_r \otimes h_r e_1^r \cdots e_{n_r}^r h_r'.$$

Since E is the infinite dimensional Grassmann algebra, we can choose homogeneous elements h_i, h_i', t_i, e_i^j in E such that

$$h_1 e_1^1 \cdots e_{n_1}^1 h_1' t_1 h_2 e_1^2 \cdots e_{n_2}^2 h_2' t_2 \cdots t_{r-1} h_r e_1^r \cdots e_{n_r}^r h_r' \neq 0. \tag{9}$$

Hence, by (8) and (9) the polynomial $u_1 f_1 v_1 w_1 u_2 f_2 v_2 w_2 \cdots w_{r-1} u_r f_r v_r$ takes a non-zero value and the proof of the lemma is complete. □

A main tool in what follows is a technique of gluing Young tableaux given in [12] that we now describe.

Let $\lambda^1 \vdash n_1, \dots, \lambda^k \vdash n_k$ be partitions with the property that for every $i = 1, \dots, k$,

$$h(d_i, l_i, t_i - s_i) \leqslant \lambda^i \leqslant h(d_i, l_i, t_i), \tag{10}$$

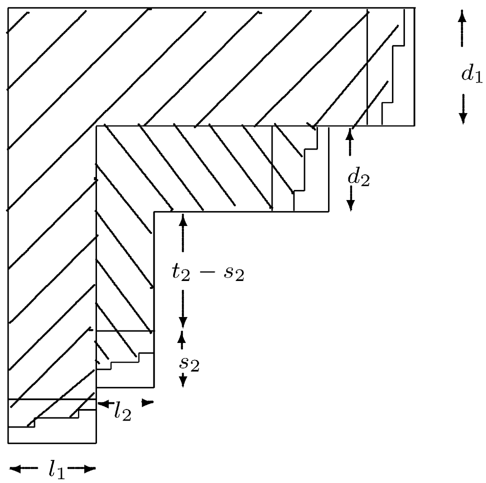
and, for every $i = 1, \dots, k-1$,

$$t_i - s_i \geqslant \max\{t_{i+1} + l_{i+1}, t_{i+1} + d_{i+1}\}, \tag{11}$$

where d_i, l_i, t_i, s_i, $1 \leqslant i \leqslant k$, are fixed integers.

As usual we identify a partition with the corresponding Young diagram. Then, by the above relations, we can glue the 1st row of λ^{i+1} to the (d_i+1)th row of λ^i, the 2nd row of λ^{i+1} to the (d_i+2)th row of λ^i and so on. In this way we get a new partition $\lambda^i \star \lambda^{i+1}$ of the integer $n_i + n_{i+1}$.

For instance the picture below illustrates the gluing $\lambda^1 \star \lambda^2$ of the partitions λ^1 and λ^2.



Notice that, if $\mu = \lambda^1 \star \cdots \star \lambda^k$ denotes the partition obtained by gluing together $\lambda^1, \dots, \lambda^k$ as above, then

$$h(d, l, t_k - s_k) \leqslant \mu \leqslant h(d, l, t)$$

where $d = d_1 + \cdots + d_k$, $l = l_1 + \cdots + l_k$ and $t \geqslant \max\{t_1 + l_1 - l, t_1 + d_1 - d\}$.

We can also glue Young tableaux in a similar way: let $T_{\lambda^1}, \dots, T_{\lambda^k}$ be Young tableaux corresponding to the partitions $\lambda^1, \dots, \lambda^k$, respectively. Define new tableaux by the following rule: set $T'_{\lambda^1} = T_{\lambda^1}$ and, for $i = 2, \dots, k$, let T'_{λ^i} be the Young tableau obtained from T_{λ^i} by adding the integer $n_1 + \cdots + n_{i-1}$ to all entries of T_{λ^i}. Then we denote by

$$T_\mu = T'_{\lambda^1} \star \cdots \star T'_{\lambda^k}$$

the Young tableau obtained by gluing together the tableaux $T'_{\lambda^1}, \dots, T'_{\lambda^k}$ (we glue the corresponding diagrams by the above procedure and we insert the integers of the tableaux $T'_{\lambda^1}, \dots, T'_{\lambda^k}$). It is clear that the tableau so obtained is in the distinct entries $1, 2, \dots, n$, where $n = n_1 + \cdots + n_k$. Moreover, by [12, Lemma 14],

$$e_{T_\mu} = e_{T'_{\lambda^1}} \cdots e_{T'_{\lambda^k}} + b \tag{12}$$

where $b \in \operatorname{span}\{\sigma \in S_n \mid \sigma(N_i) \nsubseteq N_i \text{ for some } 1 \leqslant i \leqslant k\}$, where N_i denotes the set of integers in the tableau T'_{λ^i}.

Next we shall apply the above gluing technique to multitableaux. Consider multipartitions

$$\langle\lambda^j\rangle = \big(\lambda^j(1),\dots,\lambda^j(s)\big) \vdash \big(n^j(1),\dots,n^j(s)\big),$$

with $1 \leqslant j \leqslant r$. Suppose that for each $i = 1,\dots,s$, the partitions $\lambda^1(i),\dots,\lambda^r(i)$ satisfy conditions (10) and (11) above. Then we can glue these partitions obtaining in this way a new multipartition $\langle\mu\rangle = (\mu(1),\dots,\mu(s))$, where $\mu(i) = \lambda^1(i) \star \cdots \star \lambda^r(i)$, $1 \leqslant i \leqslant s$.

We denote $n_0 = 0$, $n^0(i) = 0$ and $n_i = n^1(i) + \cdots + n^r(i)$, $1 \leqslant i \leqslant s$, and let $T_{\langle\lambda^j\rangle} = (T_{\lambda^j(1)},\dots,T_{\lambda^j(s)})$, $1 \leqslant j \leqslant r$, be a multitableau corresponding to the multipartition $\langle\lambda^j\rangle$. If we add $n_0 + n_1 + \cdots + n_{i-1} + n^0(i) + n^1(i) + \cdots + n^{j-1}(i)$ to all entries of $T_{\lambda^j(i)}$, $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant r$, we get new multitableaux which we denote by $T'_{\langle\lambda^j\rangle} = (T'_{\lambda^j(1)},\dots,T'_{\lambda^j(s)})$.

Then we define

$$T_{\langle\mu\rangle} = T'_{\langle\lambda^1\rangle} \star \cdots \star T'_{\langle\lambda^r\rangle} = \big(T'_{\lambda^1(1)} \star \cdots \star T'_{\lambda^r(1)},\dots,T'_{\lambda^1(s)} \star \cdots \star T'_{\lambda^r(s)}\big),$$

and for $1 \leqslant j \leqslant r$, $1 \leqslant i \leqslant s$, set

$$N^j(i) = \big\{n_0 + \cdots + n_{i-1} + n^0(i) + \cdots + n^{j-1}(i) + 1,\dots,n_0 + \cdots + n_{i-1} + n^0(i) + \cdots + n^j(i)\big\}.$$

Also, set

$$N(i) = N^1(i) \cup \cdots \cup N^r(i) = \{n_0 + \cdots + n_{i-1} + 1,\dots,n_0 + \cdots + n_i\}.$$

Notice that $N^j(i)$ is the set of integers filling up the tableau $T'_{\lambda^j(i)}$, $1 \leqslant j \leqslant r$, $1 \leqslant i \leqslant s$.

An obvious application of (12), gives the following.

Lemma 12. *Let $\langle\lambda^j\rangle = (\lambda^j(1),\dots,\lambda^j(s))$, $1 \leqslant j \leqslant r$, be multipartitions and suppose that for every i, $1 \leqslant i \leqslant s$, the partitions $\lambda^1(i),\dots,\lambda^r(i)$ satisfy conditions* (10) *and* (11)*. Also, let $T_{\langle\lambda^j\rangle} = (T_{\lambda^j(1)},\dots,T_{\lambda^j(s)})$ be multitableaux corresponding to $\langle\lambda^j\rangle$. If $T_{\langle\mu\rangle} = T'_{\langle\lambda^1\rangle} \star \cdots \star T'_{\langle\lambda^r\rangle}$ then*

$$e_{T_{\langle\mu\rangle}} = e_{T'_{\lambda^1(1)}} \cdots e_{T'_{\lambda^r(1)}} \cdots e_{T'_{\lambda^1(s)}} \cdots e_{T'_{\lambda^r(s)}} + \gamma$$

where $\gamma \in \operatorname{span}\{\sigma \in S_{n_1} \times \cdots \times S_{n_s} \mid \sigma(N^j(i)) \nsubseteq N^j(i)$, for some $1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant r\}$.

In the next lemma we apply the above gluing technique in order to construct a special non-identity of $E(A)$.

Lemma 13. *Let A be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra over an algebraically closed field F. Let $B_1,\dots,B_r$ be distinct $G \times \mathbb{Z}_2$-graded simple subalgebras of A such that $B_1 J B_2 J \cdots J B_r \neq 0$. Set $d_i = \dim(B_1 \oplus \cdots \oplus B_r)_{(g_i,0)}$, $l_i = \dim(B_1 \oplus \cdots \oplus B_r)_{(g_i,1)}$, $1 \leqslant i \leqslant s$. Then for any positive integer $t \geqslant 2r \dim A$ there exist an integer n, a multipartition*

$$\langle\lambda\rangle = \big(\lambda(1),\dots,\lambda(s)\big)$$

of n with the property that

$$h(d_i, l_i, 2t - 4\dim A) \leqslant \lambda(i) \leqslant h(d_i, l_i, 2t),$$

$1 \leqslant i \leqslant s$, *and a multitableau $T_{\langle\lambda\rangle}$, such that* $e_{T_{\langle\lambda\rangle}} f \notin \mathrm{Id}^G(E(A))$, *for some multilinear polynomial f with* $\deg f \leqslant n + 3\dim A$.

Proof. For $1 \leqslant j \leqslant r$, $1 \leqslant i \leqslant s$, define $p_i^j = \dim(B_j)_{(g_i,0)}$ and $q_i^j = \dim(B_j)_{(g_i,1)}$ so that

$$d_i = p_i^1 + \cdots + p_i^r \quad \text{and} \quad l_i = q_i^1 + \cdots + q_i^r.$$

By Lemma 9, for any integer $t_j \geqslant 1$ there exists a multipartition $\langle\lambda^j\rangle = (\lambda^j(1), \ldots, \lambda^j(s)) \vdash (n^j(1), \ldots, n^j(s))$ such that

$$h\big(p_i^j, q_i^j, 2t_j - u_j\big) \leqslant \lambda^j(i) \leqslant h\big(p_i^j, q_i^j, 2t_j\big), \quad 1 \leqslant i \leqslant s,$$

where $u_j = \dim B_j$, and a multitableau $T_{\langle\lambda^j\rangle} = (T_{\lambda^j(1)}, \ldots, T_{\lambda^j(s)})$ such that $g_j \notin \mathrm{Id}^G(E(B_j))$ for any multilinear polynomial g_j corresponding to $T_{\langle\lambda^j\rangle}$.

Let $t_1 = t \geqslant 2r\dim A$ be an arbitrary integer. For $2 \leqslant l \leqslant r$, define

$$r_l = u_{l-1} + \max\big\{p_1^l, \ldots, p_s^l, q_1^l, \ldots, q_s^l\big\}.$$

Also set $r_l' = r_l$ if r_l is even and $r_l' = r_l + 1$ if r_l is odd. Then define

$$2t_{l+1} = 2t_l - r_{l+1}', \quad 1 \leqslant l \leqslant r - 1.$$

Hence, for $1 \leqslant l \leqslant r$, we have

$$\begin{aligned} 2t_l - u_l &= 2t_{l+1} + r_{l+1}' - u_l \geqslant 2t_{l+1} + r_{l+1} - u_l \\ &= 2t_{l+1} + \max\big\{p_1^{l+1}, \ldots, p_s^{l+1}, q_1^{l+1}, \ldots, q_s^{l+1}\big\}. \end{aligned}$$

Since (10) and (11) are satisfied, for each i, $1 \leqslant i \leqslant s$, it is possible to glue the partitions $\lambda^1(i), \ldots, \lambda^r(i)$. In this way we obtain a multipartition $\langle\mu\rangle = (\mu(1), \ldots, \mu(s))$, where, for $1 \leqslant i \leqslant s$,

$$h(d_i, l_i, 2t_r - u_r) \leqslant \mu(i) \leqslant h(d_i, l_i, v_i), \tag{13}$$

for every $v_i \geqslant \max\{2t_1 + p_i^1 - d_i, 2t_1 + q_i^1 - l_i\}$.

Now we compute

$$\begin{aligned} 2t_1 - 2t_r &= \sum_{j=1}^{r-1}(2t_j - 2t_{j+1}) = \sum_{j=1}^{r-1} r_{j+1}' \leqslant r + \sum_{j=1}^{r-1} r_{j+1} \\ &= r + \sum_{j=1}^{r-1}\big(u_j + \max\big\{p_1^{j+1}, \ldots, p_s^{j+1}, q_1^{j+1}, \ldots, q_s^{j+1}\big\}\big) \\ &\leqslant r + 2\dim A \leqslant 3\dim A. \end{aligned}$$

Hence,

$$2t_r - u_r \geqslant 2t_1 - 3\dim A - u_r \geqslant 2t_1 - 4\dim A.$$

Thus, recalling that $t_1 = t$, the inclusions given in (13), become

$$h(d_i, p_i, 2t - 4\dim A) \leqslant \mu(i) \leqslant h(d_i, p_i, 2t), \quad 1 \leqslant i \leqslant s.$$

Let $T_{\langle\mu\rangle} = (T'_{\lambda^1(1)} \star \cdots \star T'_{\lambda^r(1)}, \ldots, T'_{\lambda^1(s)} \star \cdots \star T'_{\lambda^r(s)})$, where $T'_{\lambda^j(i)}$ is obtained, as above, by adding $n_0 + n_1 + \cdots + n_{i-1} + n^0(i) + n^1(i) + \cdots + n^{j-1}(i)$ to all entries of $T_{\lambda^j(i)}$, for $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant r$, where $n_0 = 0$, $n^0(i) = 0$ and $n_i = n^1(i) + \cdots + n^r(i)$.

For every $j = 1, \ldots, r$, denote by f_j a multilinear polynomial corresponding to the multitableau $T_{\langle\lambda^j\rangle}$, written in the new set of variables $\{x_{i,g_l} \mid i \in N^j(k), 1 \leqslant k \leqslant s\}$, where $N^j(k) = \{n_0 + \cdots + n_{k-1} + n^0(k) + \cdots + n^{j-1}(k) + 1, \ldots, n_0 + \cdots + n_{k-1} + n^0(k) + \cdots + n^j(k)\}$ (f_j corresponds to $T'_{\langle\lambda^j\rangle}$). By Lemma 11, the polynomial

$$f = u_1 f_1 v_1 w_1 u_2 f_2 v_2 w_2 \cdots w_{r-1} u_r f_r v_r,$$

where u_i, v_i, w_i are new G-graded variables, is not a G-graded identity of $E(A)$. Moreover, $\deg f = n + 3r - 1 \leqslant n + 3\dim A$, where $n = n_1 + \cdots + n_s$.

Therefore, in order to complete the proof, it is enough to show that $e_{T_{\langle\mu\rangle}} f \notin \mathrm{Id}^G(E(A))$.

Let θ be an evaluation of f giving a non-zero value and let $\theta(u_i) = \bar{u}_i \in B_i \otimes E$. By Lemma 12

$$\theta(e_{T_{\langle\mu\rangle}} f) = \theta(e_{T_{\mu(1)}} \cdots e_{T_{\mu(s)}} f) = \theta(e_{T'_{\lambda^1(1)}} \cdots e_{T'_{\lambda^r(1)}} \cdots e_{T'_{\lambda^1(s)}} \cdots e_{T'_{\lambda^r(s)}} f) + \theta(\gamma f),$$

where $\gamma \in \operatorname{span}\{\sigma \in S_{n_1} \times \cdots \times S_{n_s} \mid \sigma(N^j(i)) \nsubseteq N^j(i),\ \text{for some } 1 \leqslant i \leqslant s,\ 1 \leqslant j \leqslant r\}$.

Since $e^2_{T'_{\lambda^j(i)}} = \mu^j_i e_{T'_{\lambda^j(i)}}$, for some non-zero integer μ^j_i, and $f_i = e_{T'_{\lambda^i(1)}} \cdots e_{T'_{\lambda^i(s)}} f'_i$, we get that $e_{T'_{\lambda^i(1)}} \cdots e_{T'_{\lambda^i(s)}} f_i = \mu^i_1 \cdots \mu^i_s f_i$.

Hence

$$\theta(e_{T'_{\lambda^1(1)}} \cdots e_{T'_{\lambda^r(1)}} \cdots e_{T'_{\lambda^1(s)}} \cdots e_{T'_{\lambda^r(s)}} f) = \mu^1_1 \cdots \mu^1_s \cdots \mu^r_1 \cdots \mu^r_s \theta(f) \neq 0.$$

Now, in order to complete the proof we show that every summand in $\theta(\gamma f)$ is equal to zero in $E(A)$.

In fact, let $\sigma \in S_{n_1} \times \cdots \times S_{n_s}$ be such that $\sigma(N^j(i)) \nsubseteq N^j(i)$ for some j and i, $1 \leqslant j \leqslant r$, $1 \leqslant i \leqslant s$. Then $\sigma(k) \in N^q(i)$ for some $k \in N^j(i)$, $q \neq j$; this says that $\theta(\sigma f_j)$ belongs to $E(B_q)$. But then, since $\bar{u}_j \theta(\sigma f_j) \in E(B_j)E(B_q) \subseteq B_j B_q \otimes E = 0$, we have

$$\theta(\sigma f) = \bar{u}_1 \theta(\sigma f_1) \bar{v}_1 \bar{w}_1 \cdots \theta(\sigma f_k) \bar{v}_k = 0.$$

Therefore $\theta(\gamma f) = 0$ and the proof is complete. $\square$

As a corollary of the previous result we get

Lemma 14. *Let A be a finite dimensional $G \times \mathbb{Z}_2$-graded algebra over an algebraically closed field F of characteristic zero and let $p \geqslant 0$ be defined as in* (4). *Then there exist constants C_1, r_1, depending only on* $\dim A$ *with $C_1 \neq 0$, such that*

$$c_n^G\big(E(A)\big) \geqslant C_1 n^{r_1} p^n.$$

Proof. Recall that $A = B + J$ where B is a maximal $G \times \mathbb{Z}_2$-graded semisimple subalgebra of A. Now, let $B_1, \ldots, B_r$ be distinct $G \times \mathbb{Z}_2$-graded simple subalgebras of B such that

$$B_1 J B_2 J \cdots B_{r-1} J B_r \neq 0$$

and $\dim(B_1 \oplus \cdots \oplus B_r) = p = p(A)$.

Let $m = \dim A$, $d_i = \dim(B_1 \oplus \cdots \oplus B_r)_{(g_i,0)}$ and $l_i = \dim(B_1 \oplus \cdots \oplus B_r)_{(g_i,1)}$, $1 \leqslant i \leqslant s$.

Let N be any integer such that $N > (4+s)rm^2 + 5m$. If we divide $N - r(d_1 l_1 + \cdots + d_s l_s) - 3m$ by $2p$, we can write $N = 2pt + r(d_1 l_1 + \cdots + d_s l_s) + 3m + v$ for some $t \geqslant 2rm$ and $0 \leqslant v < 2p$. By Lemma 13, there exist an integer n, a multipartition

$$\langle\lambda\rangle = \big(\lambda(1), \ldots, \lambda(s)\big) \vdash (n_1, \ldots, n_s)$$

of n with the property that

$$h(d_i, l_i, 2t - 4m) \leqslant \lambda(i) \leqslant h(d_i, l_i, 2t),$$

$1 \leqslant i \leqslant s$, and a multitableau $T_{\langle\lambda\rangle}$, such that $e_{T_{\langle\lambda\rangle}} f \notin \mathrm{Id}^G(E(A))$, for some multilinear polynomial f with $n \leqslant c = \deg f \leqslant n + 3\dim A$. Notice that $n < N$.

We construct a polynomial $f' = f x_{c+1} \cdots x_N$, where $x_{c+1}, \ldots, x_N$ are new G-graded variables distinct from the ones appearing in f. It is easy to see that still $e_{T_{\langle\lambda\rangle}} f' \notin \mathrm{Id}^G(E(A))$. Now, by the branching theorem (see [15]), it follows that there exists a multipartition

$$\langle\mu\rangle = \big(\mu(1), \ldots, \mu(s)\big) \vdash (N_1, \ldots, N_s)$$

of N with $N_1 \geqslant n_1, \ldots, N_s \geqslant n_s$, i.e., $\langle\mu\rangle \geqslant \langle\lambda\rangle$, and a multitableau $T_{\langle\mu\rangle}$ such that $F S_N e_{T_{\langle\mu\rangle}} f' \notin \mathrm{Id}^G(E(A))$.

Since for $1 \leqslant i \leqslant s$, $|h(d_i, l_i, 2t - 4m)| = d_i l_i + 2(d_i + l_i)t - 4(d_i + l_i)m$, we have

$$\begin{aligned} N - \sum_{i=1}^{s} \big|h(d_i, l_i, 2t - 4m)\big| &= 2pt + r\left(\sum_{i=1}^{s} d_i l_i\right) + 3m + v - \sum_{i=1}^{s} d_i l_i - 2pt + 4pm \\ &\leqslant (r-1)\left(\sum_{i=1}^{s} d_i l_i\right) + 3m + 2p + 4pm = K, \end{aligned}$$

and K is a constant depending only on the structure of A. Since for all i, $1 \leqslant i \leqslant s$, we have that $N_i \geqslant n_i \geqslant |h(d_i, l_i, 2t - 4m)|$, it follows that $N_i - |h(d_i, l_i, 2t - 4m)| \leqslant K$, a constant. Therefore, since $\mu(i) \vdash N_i$, by Lemma 2

$$d_{\mu(i)} \geqslant N_i^{-2K} d_{h(d_i, l_i, 2t-4m)}, \quad 1 \leqslant i \leqslant s.$$

Moreover by [6], $d_{h(d_i,l_i,2t-4m)} \simeq a_i r_i^{b_i}(d_i+l_i)^{r_i}$, where $r_i = |h(d_i,l_i,2t-4m)|$.

We can now get an estimate of $c^G_{N_1,\dots,N_s}(E(A))$. In fact, by the above,

$$\begin{aligned} c^G_{N_1,\dots,N_s}\big(E(A)\big) &\geqslant d_{\mu(1)}\cdots d_{\mu(s)} \geqslant N_1^{-2K} d_{h(d_1,l_1,2t-4m)}\cdots N_s^{-2K} d_{h(d_s,l_s,2t-4m)} \\ &\geqslant \alpha N_1^{K_1}(d_1+l_1)^{r_1}\cdots N_s^{K_s}(d_s+l_s)^{r_s} \\ &\geqslant N^u (d_1+l_1)^{2t(d_1+l_1)-4m(d_1+l_1)}\cdots(d_s+l_s)^{2t(d_s+l_s)-4m(d_s+l_s)}, \end{aligned}$$

for some constants $\alpha, K_1,\dots,K_s,u$.

Let $p_i = d_i + l_i$, $1 \leqslant i \leqslant s$. Then, since $N_i \geqslant (2t-4m)p_i$, we get

$$\frac{N!}{N_1!\cdots N_s!} \geqslant \frac{((2t-4m)p_1+\cdots+(2t-4m)p_s)!}{((2t-4m)p_1)!\cdots((2t-4m)p_s)!}.$$

Thus

$$\begin{aligned} c^G_N\big(E(A)\big) &\geqslant \binom{N}{N_1,\dots,N_s} c^G_{N_1,\dots,N_s}\big(E(A)\big) \\ &\geqslant N^u \frac{((2t-4m)p_1+\cdots+(2t-4m)p_s)!}{((2t-4m)p_1)!\cdots((2t-4m)p_s)!} p_1^{(2t-4m)p_1}\cdots p_s^{(2t-4m)p_s}. \end{aligned}$$

Now recall Stirling formula (see [19]):

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{\theta_n}{12n}},$$

for some $0 \leqslant \theta_n \leqslant 1$. Then we get

$$\begin{aligned} &c^G_N\big(E(A)\big) \\ &\quad\geqslant N^u \frac{((2t-4m)p_1+\cdots+(2t-4m)p_s)^{((2t-4m)p_1+\cdots+(2t-4m)p_s)}}{((2t-4m)p_1)^{((2t-4m)p_1)}\cdots((2t-4m)p_s)^{((2t-4m)p_s)}} p_1^{(2t-4m)p_1}\cdots p_s^{(2t-4m)p_s} \\ &\quad= N^u(p_1+\cdots+p_s)^{(2t-4m)p_1+\cdots+(2t-4m)p_s} = aN^\alpha(p_1+\cdots+p_s)^N = aN^\alpha p^N, \end{aligned}$$

with constants a and α. □

Putting together Lemma 7 and Lemma 14 we get

Theorem 1. *Let A be a finite dimensional $G\times\mathbb{Z}_2$-graded algebra over an algebraically closed field of characteristic zero. Then there exist constants $C_1>0$, C_2, k_1, k_2 such that*

$$C_1 n^{k_1} p^n \leqslant c^G_n\big(E(A)\big) \leqslant C_2 n^{k_2} p^n,$$

where p equals the dimension of a suitable semisimple $G\times\mathbb{Z}_2$-graded subalgebra of A.

Theorem 2. *Let A be a G-graded PI-algebra over any field F of characteristic zero. Then there exist constants $C_1 > 0$, C_2, k_1, k_2 such that*

$$C_1 n^{k_1} p^n \leqslant c_n^G(A) \leqslant C_2 n^{k_2} p^n.$$

Hence $\exp^G(A) = \lim_{n\to\infty} \sqrt[n]{c_n^G(A)}$ *exists and is an integer.*

Proof. If K is an extension field of F then it is not difficult to prove that $c_n^G(A) = c_n^G(A \otimes_F K)$ (see [14]). Therefore we may assume that F is algebraically closed. By [20] there exists a finite dimensional $G \times \mathbb{Z}_2$-graded algebra B over F such that $\mathrm{Id}^G(A) = \mathrm{Id}^G(E(B))$; hence the conclusion now follows from Theorem 1 since $C_1 n^{k_1} p^n \leqslant c_n^G(E(B)) \leqslant C_2 n^{k_2} p^n$, for some constants C_1, C_2, k_1, k_2 and $\exp^G(E(B)) = p$. $\square$

References

[1] E. Aljadeff, A. Kanel-Belov, Representability and Specht problem for G-graded algebras, preprint.

[2] E. Aljadeff, A. Giambruno, D. La Mattina, Graded polynomial identities and exponential growth, J. Reine Angew. Math., in press.

[3] Y.A. Bahturin, V. Drensky, Graded polynomial identities of matrices, Linear Algebra Appl. 357 (2002) 15–34.

[4] Y.A. Bahturin, S.K. Sehgal, M.V. Zaicev, Finite-dimensional simple graded algebras, Sb. Math. 199 (7) (2008) 965–983.

[5] F. Benanti, A. Giambruno, M. Pipitone, Polynomial identities on superalgebras and exponential growth, J. Algebra 269 (2003) 422–438.

[6] A. Berele, A. Regev, Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras, Adv. Math. 64 (1987) 118–175.

[7] A. Berele, A. Regev, Exponential growth for codimensions of some p.i. algebras, J. Algebra 241 (2001) 118–145.

[8] C.W. Curtis, I. Reiner, Representation Theory of Finite Groups and Associative Algebras, Wiley Classics Lib., John Wiley & Sons, Inc., New York, 1988.

[9] A. Giambruno, S. Mishchenko, M. Zaicev, Codimensions of algebras and growth functions, Adv. Math. 217 (2008) 1027–1052.

[10] A. Giambruno, A. Regev, Wreath products and P.I. algebras, J. Pure Appl. Algebra 35 (1985) 133–149.

[11] A. Giambruno, M. Zaicev, On codimension growth of finitely generated associative algebras, Adv. Math. 140 (1998) 145–155.

[12] A. Giambruno, M. Zaicev, Exponential codimension growth of P.I. algebras: an exact estimate, Adv. Math. 142 (1999) 221–243.

[13] A. Giambruno, M. Zaicev, Codimension growth and minimal superalgebras, Trans. Amer. Math. Soc. 355 (2003) 5091–5117.

[14] A. Giambruno, M. Zaicev, Polynomial Identities and Asymptotic Methods, Mathematical Surveys and Monographs, vol. 122, AMS, Providence, RI, 2005.

[15] G. James, A. Kerber, The Representation Theory of the Symmetric Group, Encyclopedia Math. Appl., vol. 16, Addison–Wesley, London, 1981.

[16] A.R. Kemer, Ideals of Identities of Associative Algebras, Transl. Math. Monogr., vol. 87, AMS, 1988.

[17] V. Petrogradsky, On complexity functions for T-ideals of associative algebras, Mat. Zametki 68 (6) (2000) 887–897; Translation in Math. Notes 68 (5–6) (2000) 751–759.

[18] A. Regev, Existence of identities in $A \otimes B$, Israel J. Math. 11 (1972) 131–152.

[19] H. Robbins, A remark on Stirling's formula, Amer. Math. Monthly 62 (1955) 26–29.

[20] I. Sviridova, Identities of PI-algebras graded by a finite abelian group, Comm. Algebra, in press.

[21] E. Taft, Orthogonal conjugacies in associative and Lie algebras, Trans. Amer. Math. Soc. 113 (1964) 18–29.

[22] M. Zaicev, S. Mishchenko, An example of a variety of Lie algebras with a fractional exponent, Algebra, 11, J. Math. Sci. (N. Y.) 93 (1999) 977–982.

[23] M. Zaicev, Integrality of exponents of growth of identities of finite-dimensional Lie algebras, Izv. Ross. Akad. Nauk Ser. Mat. 66 (2002) 23–48; English translation: Izv. Math. 66 (2002) 463–487.